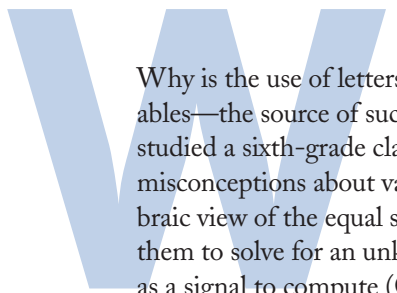


For these sixth graders, transitioning from arithmetic to algebraic thinking involved developing new meanings for symbols in expressions and equations.

Frustration WITH Understanding VARIABLES IS NATURAL

Diana L. Moss, Jennifer A. Czocher, and Teruni Lamberg



Why is the use of letters in algebraic expressions and equations—variables—the source of such uncertainty for students and teachers? We studied a sixth-grade classroom and observed that students hold many misconceptions about variables (Moss 2014). Some students hold an algebraic view of the equal sign. For them, it indicates an equation and signals them to solve for an unknown variable (Usiskin 1999). Others view it only as a signal to compute (Carpenter, Franke, and Levi 2003). As we battled our own frustrations with students' misconceptions about variables, we discovered that their meanings for variables could be further sorted into groups and that those meanings changed over time. We also found that certain problem contexts and teacher questions could help shift students' meanings of variable. In this article, we share our findings and some productive ways to work with students' natural ways of thinking.

BACKGROUND

The transition from arithmetic to algebra can be challenging for middle school students. Algebra involves “writing, interpreting, and using expressions and equations” (CCSSI 2010, p. 39). Therefore, learning to use and



Table 1 The five meanings of variables, the problem context that elicited them, and a sample of student work are shown.

Student Meanings of Letters and Variables	Description	Example of a Task with Context	Expression or Equation from Student Work
Letter as a label	The letter is used as a label to identify a category and keep track of the number of items in that category. The letter itself represents one unit. The coefficient represents the number of items in that category.	Write an expression for the number of hexagons and pentagons on a soccer ball.	$20h + 12p$, which means 20 hexagons + 12 pentagons
Variable as a changing quantity	The variable is used to represent different or changing values.	Write an expression to determine the total cost of purchasing 4 packs of 6 cupcakes and a single cupcake. The packs of cupcakes and the single cupcake have different prices depending on different stores.	$4s + c$, which means the cost of 4 packages of 6 cupcakes + the cost of a single cupcake
Variable as a known value	In a known-quantity variable, the value of the variable is given.	The cost of 4 packages of 6 cupcakes is \$6 and the cost of a single cupcake is \$1. Using the expression for the cost of cupcakes ($4p + s$), substitute the given price for each variable to determine the total cost.	$s = 6$ $c = 1$ $4s + c = 4(6) + 1$ $= 24 + 1$ $= 25$ The total cost is \$25.
Variable as an unknown quantity	The variable is the unknown value in an equation.	Max has a certain number of soccer balls, and David gives him 4 more soccer balls. Now Max has 8 soccer balls. How many did Max have in the first place?	b is the number of soccer balls that Max has. $b + 4 = 8$ $- 4 \quad - 4$ $b = 4$
Variables as independent or dependent	One variable in an equation is the independent variable, and the other variable is the dependent variable. There is a relationship between the independent and dependent variable.	Write an equation for the perimeter of a square.	p is perimeter s is length of a side of the square $p = 4s$

interpret variables is at the center of the transition. Adding to the confusion is the fact that algebraic variables can be used in many different ways (Philipp 1992). Variables can be seen as labels for quantities (Booth 1988; Clement 1982; Weinberg et al. 2004) or as letters that represent varying quantities rather than specific values (Stephens 2005).

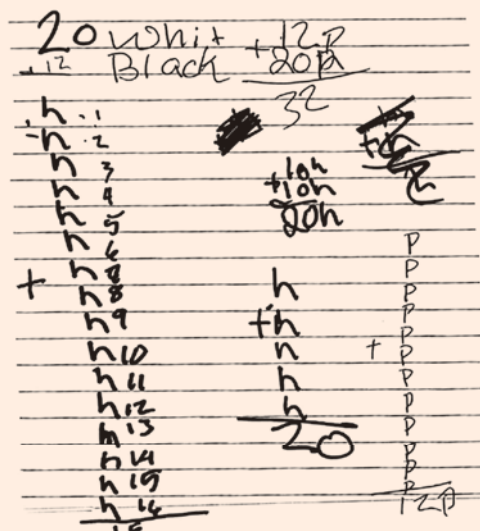
Research indicates that students

must not only understand multiple meanings of variables but also develop flexible meanings of the equal sign to make sense of algebraic expressions and equations (Rosnick 1981; Schoenfeld and Arcavi 1988; Philipp 1992). The context of the problem determines how the variable and the equal sign should be interpreted (Lappan 2000). Mathematically, a variable represents a quantity symbolized by a letter in an

algebraic expression or equation. Problems must be carefully selected to help students develop appropriate meanings of variable.

As students incorporate new concepts of algebra into their existing knowledge, some misconceptions naturally arise in their learning process, such as thinking that a variable is only a letter that abbreviates a word or represents a number that does not

Fig. 1 Anna's work (a) and her discussion with Ms. J. (b) demonstrate thinking about a letter as a label.



(a)

Anna: I have 16 *hs* plus 4 *hs*, and that equals twenty.

Ms. J: Twenty what?

Anna: Twenty *hs*.

Ms. J: Which represents? What does *h* mean?

Anna: Hexagons.

Ms. J: Why did you write it twenty times? How many hexagons are on the soccer ball?

Anna: Twenty.

Ms. J: If I write a single *h*, what does that represent?

Anna: One hexagon.

Ms. J: So, to show twenty hexagons, we write?

Anna: [We write] $20h$.

(b)

vary. The next section describes the different meanings of variables that arose naturally as we worked in a sixth-grade classroom. We then share some ways that teachers can help students understand algebraic variables and the equal sign.

MEANINGS FOR VARIABLES REVEALED BY STUDENT THINKING

To see how middle school students made sense of variables and the equal sign, we worked with a teacher, Ms. J., to develop lessons for the sixth-grade expressions and equations strand of CCSSM (CCSSI 2010). The lessons took place over four weeks and were designed to help students transition from arithmetic to algebraic reasoning with a focus on operations and strategies that are applied to known and unknown quantities (Moseley and Brenner 2009). Throughout the instructional unit, we documented five meanings of variables (Moss 2014) in students' written work and in their discussions with their teacher that provided us with insight into how their thinking developed. **Table 1** describes

the five meanings and includes examples of the tasks and student work.

SAMPLES OF STUDENTS' NATURAL MEANINGS FOR VARIABLES

Letter as a Label

We use the term *letter* to describe this kind of thinking instead of *variable* because in this context, students do not perceive the symbol to be unknown or changing, and, thus, it is not a variable. Students used letters as units to represent a category (much like 5s can represent 5 seconds). Using a letter as a label helped students identify the category and keep track of specific units when counting and combining like and unlike terms. The coefficients are used multiplicatively to keep track of how many items are in the category (its magnitude). This is the essence of thinking about a letter as a label.

For example, students were given miniature soccer balls and asked to count the number of hexagons and the number of pentagons and then to write an expression for the total number of polygons. The purpose of this task was

to help students explore combining like terms and realize that unlike terms cannot be combined. This activity provided students with the opportunity to discover that items in the same category can be added to find the total. One student, Anna, used the letter *h* to keep track of a tally of hexagons (see **fig. 1**).

Anna used an arithmetic approach. She recorded each hexagon and pentagon in a vertical list of *hs* and *ps*. She added the total number of pentagons and the total number of hexagons and then wrote the expression $12p + 20h = 32$ for the total number of polygons. Ms. J. realized that Anna was reasoning as though *p* and *h* had no value and then viewing the equal sign as a signal to compute the sum of the coefficients. In Anna's work, the letters *h* and *p* functioned as labels for the items in the categories *hexagon* and *pentagon*. The numbers in the first column next to each letter represented a cumulative total of counting in a sequence.

Anna concluded that 16 *hs* were evident at the bottom of the addition problem (see **fig. 1a**). An additional column with four *hs* was created to represent the remaining hexagons in

Ways to Support Student Thinking

1. Encourage student thinking about variables and the equal sign:

Students need to explicitly think about and discuss the meaning of variables in various problem contexts. Transitioning from seeing the equal sign as a symbol to find an unknown value toward seeing it as a relation between two equal quantities is also critical. This involves carefully selecting tasks and considering how the problem context lends itself to reasoning about variables (e.g., see the tasks in **table 1**). To support student thinking about variables, it is important for teachers to prompt students to draw on their real-world experiences with varying quantities (Moss 2014).

2. Pose questions that scaffold student thinking:

Questions such as those listed below can help students reflect on the meanings of variables and the equal sign. These questions can reveal valuable information about how the student is thinking and so help build on earlier conceptions of variable.

What does the variable represent to you?

- Helps students focus on how they are using the variable (e.g. to label a single polygon, all of the polygons, to count the number of sides of a polygon)
- Engages students in discussion about their meaning for the variable

What stays the same and what changes?

- Draws students' attention to the idea that a variable is a changing quantity (e.g. the price of the cupcakes)
- Assists students with connecting a realistic context with an algebraic expression or equation

What does the equal sign mean?

- Focuses students on equal sign as “equality” rather than a symbol to “compute”
- Supports students in coming to understand variables as unknown values

What value does the variable have?

- Supports students to substitute a known value for the variable and helps them to understand independent and dependent variables.

3. Accept that misconceptions are a natural part of learning:

The way that students come to understand expressions and equations results in specific misconceptions about variables and the equal sign that are natural for the learning process. These misconceptions can be used to modify instruction and scaffold student learning in meaningful ways. Students naturally thought about letter as a label because arithmetic thinking lends itself to finding sums and differences among categories. For example, students are often asked to find the total number of apples and bananas in a fruit basket and might think that a is for apples and b is for bananas, just as our students labeled pentagons with p and hexagons with h on a soccer ball. In the Cupcake problem, the teacher allowed students to label a package of 6 cupcakes with s and 1 cupcake with c to write the equation $4s + c$ and then made it explicit that this expression represented the cost of the cupcakes where s and c were quantities. We conclude that the earlier ways of thinking about variables need not be avoided or circumvented; they are important understandings of the meaning of variables that support later, more sophisticated meanings.

the soccer ball. At this point, Anna did not write the cumulative total. Instead, she counted and kept track mentally. She indicated that the total was 20 and clarified her thinking in the whole-class discussion that took place with Ms. J. (see **fig. 1b**). Anna shared her drawing and explained that she had 16 h s plus 4 h s and gestured to the work in her notebook to illustrate that there were

20 hexagons in total. Therefore, it is evident that she was thinking that each h represented a single unit. Also, when probed by the teacher, Anna explained that the h represented 1 hexagon.

Tasks like the Soccer Ball problem allow a letter to represent a category. Therefore, many students spontaneously used letter notation as a label to keep track of their counting. They could use

familiar arithmetic thinking to count items in each category while beginning to use algebraic thinking to combine like terms. Real-world contexts, such as that in the Soccer Ball problem, that require combining items in the same category are helpful for exploring like and unlike terms, an important precursor to reasoning about variables as unknown and changing quantities.

Fig. 2 This task, the Cupcake problem, shifted students' thinking about variables

The Cupcake Problem

We have 24 students in the class plus the teacher, which means we need 25 cupcakes so that everyone in the class can have one. Cupcakes come in packages of 6 or you can buy a single cupcake. Find an algebraic expression that describes the total cost of cupcakes.

Fig. 3 This work demonstrates the class interpreting a variable as a changing quantity (a) and Alan interpreting a variable as a known value (b).

24 students
1 teacher = 25 total

Cupcakes come in 6
a package of six
or you can buy a single
cupcake. 1

We need a formula
for the cost of the
cupcakes.

$4s + c$

(a)

Handwritten work for Figure 3(b) showing calculations for different stores:

- At Walmart: $4p + 1s$, $p = \$6.00$, $s = \$1.00$, $4(\$6.00) + \$1.00 = \$24.00 + \$1.00 = \$25.00$
- At Safeway: $4p + 1s$, $p = \$8.00$, $s = \$2.00$, $4(\$8.00) + \$2.00 = \$32.00 + \$2.00 = \$34.00$
- At Raley's: $4p + 1s$, $p = \$7.00$, $s = \$1.00$, $4(\$7.00) + \$1.00 = \$28.00 + \$1.00 = \$29.00$
- At Sak-n-Save: $4p + 1s$, $p = \$4.50$, $s = \$0.75$, $4(\$4.50) + \$0.75 = \$18.00 + \$0.75 = \$18.75$

(b)

Variable as a Changing Quantity, and Variable as a Known Value

The Cupcake problem (see **fig. 2**) shifted student thinking about variable toward seeing variables as representing known values or changing quantities because the exact prices of the cupcakes are unspecified. The entire class discussed the idea that the letters represented prices and so had some value. This helped orient students to the idea that the symbols were being used differently than in the Soccer Ball problem. The symbols were not just labeling the categories "a package of 6 cupcakes" and "a single cupcake." The class also discussed how prices might be different at different stores and that the variables in the expression $4s + c$ represented changing quantities.

As students worked through this problem in small groups, a student in one group wrote the expression $s + s + s + s + c$ to represent the prob-

lem situation. This group of students realized that the expression could be simplified to $4s + c$. Students' prior knowledge of combining like and unlike terms, such as in the Soccer Ball problem, made it easy for them to make sense of and simplify the expression. The expression $4s + c$ uses s to represent a package of 6 cupcakes and c to represent a single cupcake. Initially, it is unclear by the class work (see **fig. 3**) whether the letters represent the costs, *quantities*, or whether the letters are labeling the types of packages, *objects*. However, the final expression $4s + c$ is recorded as the total cost of the cupcakes, a quantity (see **fig. 3a**). Pursuing this idea that the variables are changing quantities, Ms. J. gave the prices for different stores (see **fig. 4**). The variables then represented known quantities (the prices at each store) because the values of the variables

Fig. 4 Ms. J. gave the prices of cupcakes at different stores.

- At Wal-Mart, one package of six cupcakes is \$6.00, and a single cupcake is \$1.00.
- At Safeway, one package of six cupcakes is \$8.00, and a single cupcake is \$2.00.
- At Raley's, one package of six cupcakes is \$7.00, and a single cupcake is \$1.00.
- At Sak-n-Save, one package of six cupcakes is \$4.50, and a single cupcake is \$0.75.

were given and the same expression could be used to represent the total cost of the cupcakes even though the cost was not fixed across all of the stores. Alan's work (see **fig. 3b**) shows how he worked with the expression as though the variables were known values. The students' new view of the expression signals a shift toward algebraic thinking.

Alan, like Anna, viewed the equal sign as a signal to compute once he substituted the known value for the variable. Unlike Anna, he was thinking about $4p$ not as 4 packages of cupcakes (where p would be a label) but rather as 4 times the price of 1 package of cupcakes (where p can vary). Alan evaluated his expression, $4p + 1s$, with the variables as known values and proceeded to find the total cost of the cupcakes using multiplication and addition. Once he substituted known values for the variables, Alan transitioned to arithmetic thinking to find the total cost of the cupcakes.

Variable as an Unknown Value

The Cleats problem (see **fig. 5**) has a context in which the variable represents a specific quantity whose value is not provided. This context shifted students' thinking toward viewing variable as an unknown value. *Variable as*

Fig. 5 Kate's work illustrated a major shift toward algebraic reasoning

The Cleats Problem	
Cleats and a ball together cost \$72. If the cleats cost \$55, what is the cost of the ball?	
Write the algebraic equation.	Solve the equation.
$x + 55 = 72$	$\begin{array}{r} 672 \quad 55 \\ -55 \quad +17 \\ \hline 17 \quad 72 \end{array}$ The cost is \$17.

an unknown value differs from variable as a changing value. In the former, the variable has only one unknown value that balances the equation. In the latter, the variable can represent different quantities. **Figure 5** shows Kate's work. She wrote an algebraic equation and solved for the unknown. She wrote $x + 55 = 72$, in which x is the cost of the ball. She viewed x as an unknown value and isolated it to find the cost of the ball. Kate solved the equation by subtracting 55 from 72 to find the unknown value for 17, which was the cost of the ball. In this context, the variable in the equation was not a changing quantity; only one value made the equation true. Kate's work illustrates a major shift toward algebraic reasoning. She did not view the equal sign as a signal to compute but rather understood that an equation maintains equality between quantities on both sides. She then used inverse operations to find the unknown value of x . Once Kate found that the unknown was 17, x became a known value and was no longer an unknown value.

Variables as Independent or Dependent

Students were asked to work in small groups to find an equation for the perimeter of a square. In a whole-class discussion, students came up with the

equation $4s = p$ for the perimeter of a square. Ms. J. continued the discussion to scaffold student learning about the meaning of variable as independent and dependent quantities that change together.

Ms. J.: [The expression] $4s = p$ is called a *function*. It has input and output values. If I input, I can input any number and get an output. So, what we are going to do with this [pointing at the $4s = p$] is we are going to create an arrow diagram.

Class: What is an arrow diagram?

Ms. J.: It is a chart to help us. When we have the input and output like we labeled there [pointing to the expression $4s + p$], we can make an arrow diagram. The inputs can be called the *domain*, and the outputs can be called the *range*. So, like our x and y when we are plotting, this is similar. If I input a 1 into my formula, what will my output be? I want you to do this in your notebooks [gesturing to have students fill out the arrow diagram].

[Students work together to fill out the arrow diagram in **fig. 6**.]

Ms. J.: I'm going to tell you what the input is, and you raise your hand if you know what the output is. My input is 1; what is my output?

Gina: 4.

Ms. J.: My input is 2; what is my

output?

Leah: 8.

Ms. J.: My input is 4; what is my output?

Jason: 12.

Ms. J.: My input is 4; what is my output?

Britney: 16.

[They continue for two more rounds.]

Ms. J.: Can everyone see why this is called an *arrow diagram*?

Class: Yes.

Ms. J.: If I were to graph this, who can raise their hand and tell me what would be my x -coordinate and what would be my y -coordinate?

Anna: x would be the inputs.

Ms. J.: What would be my y ?

John: Outputs.

Ms. J.: Good. Another way to say this is that the input can also be known as the *independent variable*. So, what might the output be called?

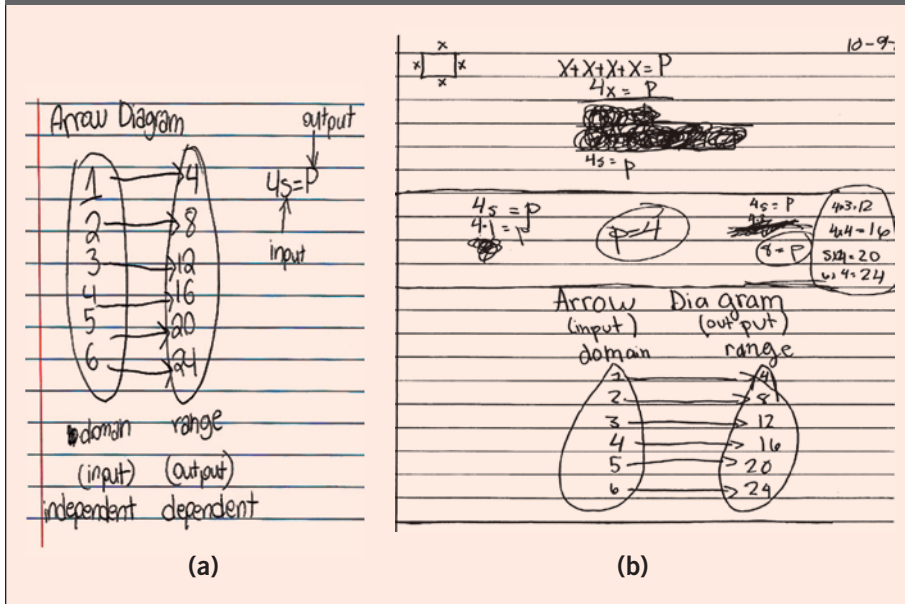
Class: Dependent.

Ms. J.: Why would this be the dependent? What does *dependent* mean?

Eric: It depends on the other numbers [pointing to the inputs in his arrow diagram].

Jason's and Eric's algebraic reasoning about independent and dependent variables drew on their prior knowledge of combining like terms, coefficients, variable as a changing quantity, and variable as a known value. In Jason's equation, $4s = P$, s is the side length of the square, and P is the perimeter. He used the first letter of each word as labels for the variables. Students viewed the variables in $4s = P$ and $x + x + x + x = P$ as changing quantities because they saw that different values could be substituted for the variables (see the arrow diagrams in **fig. 6**). When Eric substituted $s = 3$ into $4s = P$, s became a known value and P became an unknown value. This problem also encouraged students to make sense of the equal sign as a signal to balance both sides of the

Fig. 6 Jason's (a) and Eric's (b) arrow diagrams helped them make sense of the equation for the perimeter of a square by organizing the inputs and outputs.



equation. The context developed algebraic reasoning about the equal sign as signaling a relationship between the independent and dependent variables. Once the variables became known values, students substituted known values for the sides of the square to find the perimeter. At this point, the equal sign became a signal to compute, requiring arithmetic thinking.

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